University of Stuttgart
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Deep learning methods for stochastic Galerkin approximations of elliptic random PDEs.

Pathways into Mathematics of SPDEs:
A Workshop for Young Researchers

March 9 - 11, 2026

Computational Methods of Uncertainty Quantification

Field of research

- Random PDEs
- Stochastic Galerkin Approximations
- Deep Learning methods for RPDEs

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Elliptic random PDE

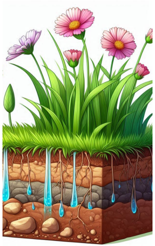
Let $(\Omega, \mathcal{A}, P)$ be a complete probability space, and let $\mathcal{D} \subset \mathbb{R}^d$, $d \in \mathbb{N}$, be an open, bounded and connected domain with Lipschitz boundary $\partial\mathcal{D}$ and closure $\overline{\mathcal{D}}$.

Strong formulation of the elliptic RPDE

Find the solution $u : \Omega \times \overline{\mathcal{D}} \to \mathbb{R}$, such that:

$$\begin{aligned} -\nabla \cdot (a \, \nabla u(\omega, x)) &= f & \quad & x \in \mathcal{D},\ \omega \in \Omega, \\ u(\omega, x) &= 0 & & x \in \partial\mathcal{D},\ \omega \in \Omega, \end{aligned}$$

where the arising differential operators are understood with respect to the spatial variable $x \in \mathcal{D}$.



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- Benchmark model in uncertainty quantification to model subsurface flows

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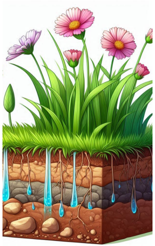

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- Benchmark model in uncertainty quantification to model subsurface flows
- Stochastic diffusion coefficient $a : \Omega \times \mathcal{D} \to \mathbb{R}$ models permeability

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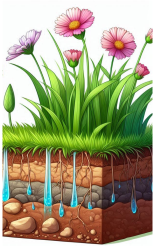

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- Stochastic forcing term $f : \Omega \times \mathcal{D} \to \mathbb{R}$ represents a source

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- Stochastic forcing term $f : \Omega \times \mathcal{D} \to \mathbb{R}$ represents a source
- Solution $u : \Omega \times \overline{\mathcal{D}} \to \mathbb{R}$ of the random PDE is a random field as well

Elliptic random PDE

Let $(\Omega, \mathcal{A}, P)$ be a complete probability space, and let $\mathcal{D}(\omega) \subset \mathbb{R}^d$ (for each $\omega \in \Omega$) be open, bounded and connected with Lipschitz boundary.

Strong random domain formulation of the elliptic PDE

Find the solution $u : \Omega \times \overline{\mathcal{D}(\omega)} \to \mathbb{R}$, such that:

$$
\begin{aligned}
-\nabla \cdot (a(x)\, \nabla u(\omega, x)) &= f(x) \qquad && x \in \mathcal{D}(\omega),\ \omega \in \Omega, \\
u(\omega, x) &= 0 && x \in \partial\mathcal{D}(\omega),\ \omega \in \Omega,
\end{aligned}
$$

where the arising differential operators are understood with respect to the spatial variable $x \in \mathcal{D}(\omega)$.

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- Benchmark model in uncertainty quantification to model subsurface flows
- Diffusion on a random domain $\mathcal{D}(\omega)$
- Solution $u : \Omega \times \overline{\mathcal{D}(\omega)} \to \mathbb{R}$ of the random domain problem is a random field as well

Solution methods of the random PDE

Random PDE

$$-\nabla \cdot (a(\omega, x)\, \nabla u(\omega, x)) = f(\omega, x)$$

Statistical approach

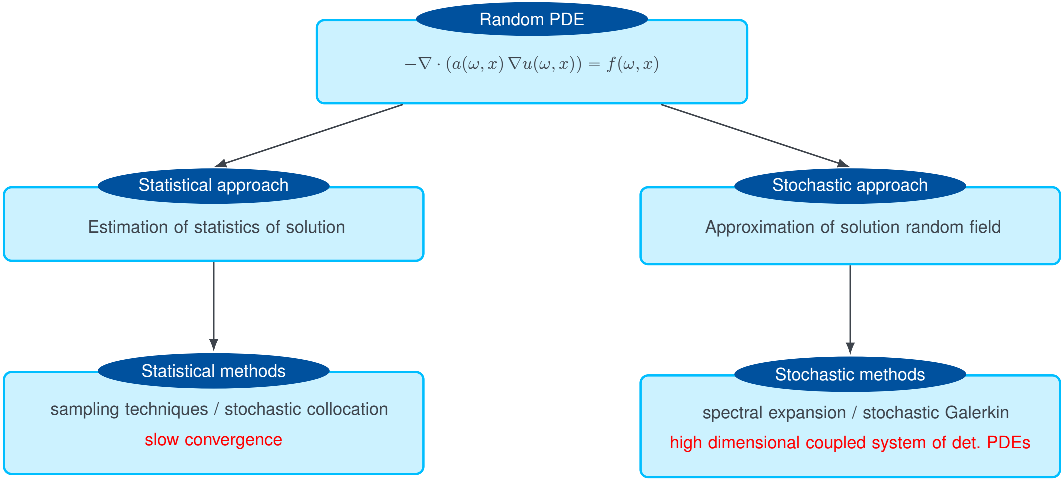
Estimation of statistics of solution

Statistical methods

sampling techniques / stochastic collocation

slow convergence

Solution methods of the random PDE



Overview: stochastic Galerkin approaches

RPDE strong formulation

$$-\nabla \cdot (a(\omega, x)\nabla u(\omega, x)) = f(\omega, x)$$

↔

RPDE weak formulation

$$\int_\Omega \int_\mathcal{D} a(\omega, x)\nabla u(\omega, x) \cdot \nabla v(\omega, x)\, dx\, dP(\omega) = \int_\Omega \int_\mathcal{D} f(\omega, x) v(\omega, x)\, dx\, dP(\omega)$$

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RPDE weak formulation

$$B(u, v) = F(v)\, \forall v \in L^2(\Omega; H(\mathcal{D}; \mathbb{R}))$$

Overview: stochastic Galerkin approaches

RPDE strong formulation

- Requires a strong solution
- First and second order derivatives appear
- Derivative of diffusion coefficient appears

RPDE weak formulation

- Rich Lax-Milgram theory
- First order derivatives appear
- No derivative of diffusion coefficient
- Testing necessary

Polynomial chaos expansion

Finite Noise

$$Y = (Y_1, \dots, Y_N) : \Omega \to \Gamma \subset \mathbb{R}^N$$

A. Mugler and H.-J. Starkloff, On the convergence of the stochastic Galerkin method for random elliptic partial differential equations, ESAIM: Mathematical Modelling and Numerical Analysis

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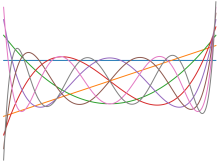
Orthonormal tensorproduct polynomial Basis
$(p_k)_k$ of $L^2(\Gamma; \mathbb{R})$



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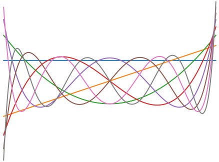

Spectral expansion

$$\phi(\omega, x) = \sum_k \phi_k(x) p_k(Y(\omega))$$
$$\phi(y, x) = \sum_k \phi_k(x) p_k(y)$$

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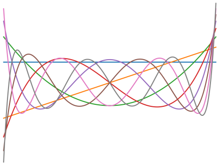

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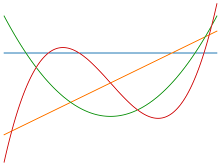
Truncation at maximal polynomial degree $P \in \mathbb{N}$



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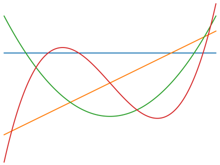

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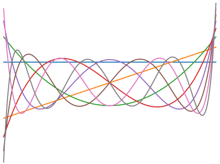

Coupling dimensionality
$M + 1 = \frac{(N+P)!}{N!P!}$

Spectral expansion
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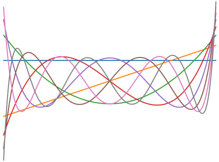
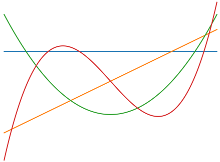


Truncation at maximal polynomial degree $P \in \mathbb{N}$



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$M + 1 = \frac{(N+P)!}{N!P!}$

Spectral expansion
$\phi(\omega, x) = \sum_k \phi_k(x) p_k(Y(\omega))$
$\phi(y, x) = \sum_k \phi_k(x) p_k(y)$

Solution space
$L^{2;(M)}(\Gamma; \mathbb{R}) \otimes H(\mathcal{D}, \mathbb{R})$
$L^{2;(M)}(\Gamma; \mathbb{R}) = \mathrm{span}(p_0, \ldots, p_M)$

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Overview: stochastic Galerkin approaches

RPDE strong formulation

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Polynomial chaos expansion
Stochastic Galerkin

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Polynomial chaos expansion
Stochastic Galerkin

Strong SGA

$$\mathcal{R}_{\bar{u}^{(M)}} = \nabla \cdot \left(\bar{a}^{(M)} \nabla \bar{u}^{(M)}\right) + \bar{f}^{(M)}$$

$$\langle \mathcal{R}_{\bar{u}^{(M)}}(\cdot, x), p_k \rangle_{L^2(\Gamma; \mathbb{R})} \stackrel{!}{=} 0$$

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Weak SRA

$$\bar{B}^{(M)}(\bar{u}^{(M)}, \bar{v}^{(M)}) = \bar{F}^{(M)}(\bar{v}^{(M)})$$

$$\forall \bar{v}^{(M)} \in L^{2;(M)}(\Gamma; H(\mathcal{D};\mathbb{R}))$$

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Weak SRA

$$\bar{\mathcal{E}}^{(M)}(\bar{u}^{(M)})$$

$$:= \frac{1}{2}\bar{B}^{(M)}(\bar{u}^{(M)},\bar{u}^{(M)}) - \bar{F}^{(M)}(\bar{u}^{(M)})$$

$$u^\star = \arg\min_{\bar{u}^{(M)}} \bar{\mathcal{E}}^{(M)}(\bar{u}^{(M)},\bar{u}^{(M)})$$

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Polynomial chaos expansion
Galerkin | Ritz

Strong SGA

- Set projected strong residual to zero
- Coupled system for spectral coefficients
- Generally applicable

Weak SRA

- Minimizing energy functional
- Coupled system for spectral coefficients
- No testing necessary
- Applicable in special cases

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Deep learning solution approach

Stochastic Galerkin method

- Curse of dimensionality: dimension of coupled system grows exponentially in model and approximation complexity
- Conventional PDE solution techniques reach computational limits fast
- Method is usually limited to low complexity or special cases that enable decoupling schemes

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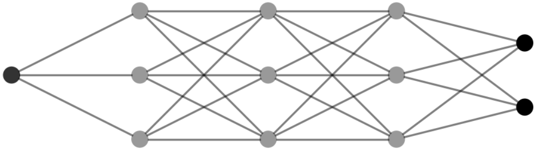
Deep learning approach to the stochastic solution

- Substitute a conventional PDE solver with a deep learning approach
- Mitigate curse of dimensionality: solved coupled system in fairly high dimensions on a single NIVIDA RTX 3070 GPU
- Cost of giving up theoretical error bounds and convergence rates

Deep learning approach

Neural network surrogate $\bar{u}^{(M)}(y,x) \approx e(x) \sum_{k=0}^{M} \mathcal{N}_{k;\theta}(x) p_k(y)$



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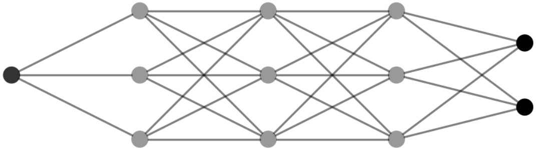

Boundary conditions enforcing
$e(x) = 0,\ x \in \partial\mathcal{D},\ e(x) > 0,\ x \in \mathcal{D}$

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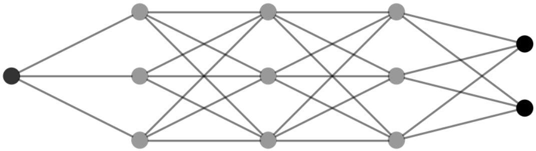

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Loss functions based on stochastic Galerkin/Ritz formulations
Training is unsupervised

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- Applicable in special cases

Deep neural network surrogate

S-GalerkinNet

- Physics-informed neural network
  (M. Raissi, P. Perdikaris, G. E. Karniadakis)
- Higher computational cost
- Higher hardware requirements

S-RitzNet

- Deep-Ritz neural network
  (W. E, B. Yu)
- Loss function given naturally

Numerical experiment: log-normal diffusion

Log-normal diffusion coefficient

Gaussian random field, covariance kernel

$k(x, x') = \exp\left(-\frac{1}{2}(x - x')^2\right)$

Truncated KLE: $a(\omega, x) = \exp\left(\sum_{k=0}^{N-1} \sqrt{\lambda_k}\phi_k(x)Y_k(\omega)\right)$

Forcing term $f \equiv 1$

Domain $\mathcal{D} = (0, 1)$

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KLE truncation after $N = 3$
Maximal polynomial dimension $P = 0,\dots,7$

Domain $\mathcal{D} = (0,1)$

Error $\epsilon_{M,\theta} \approx \frac{\left\|u-u_\theta^{(M)}\right\|_{L^2(\Omega;H^1(\mathcal{D};\mathbb{R}))}}{\|u\|_{L^2(\Omega;H^1(\mathcal{D};\mathbb{R}))}}$

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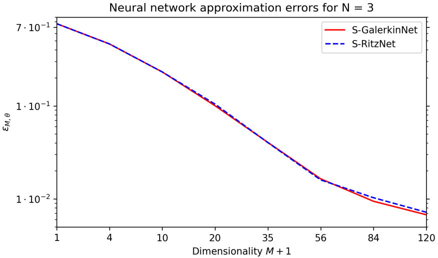
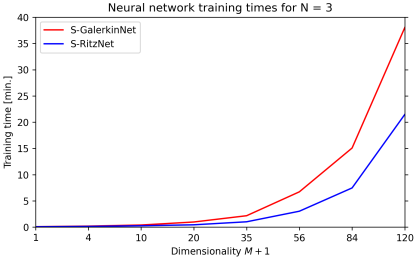
Error $\epsilon_{M,\theta} \approx \frac{\left\| u - u_\theta^{(M)} \right\|_{L^2(\Omega;H^1(\mathcal{D};\mathbb{R}))}}{\| u \|_{L^2(\Omega;H^1(\mathcal{D};\mathbb{R}))}}$





Numerical experiment: KL-type diffusion

KL-type diffusion coefficient

$$a(\omega,(x_1,x_2)) = 3 - x_1x_2(1-x_1)(1-x_2) - \frac{1}{2}\sum_{k=1}^{200}\left(\frac{1}{k}\right)^{8/5} e^{-\frac{1}{k}(x_1-x_2)^2}(Y_k(\omega)+1)$$

$$Y_k \sim \mathcal{U}([-1,1])$$

Forcing term $f \equiv 1$

Domain $\mathcal{D} = (0,1)^2$

KL-type truncation after $N = 1, \ldots, 40$

Maximal polynomial degree $P = 1$

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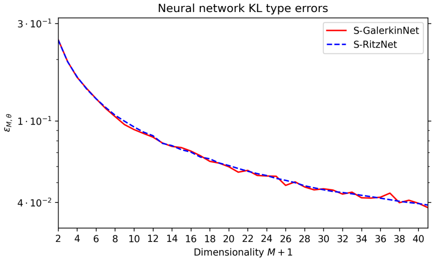
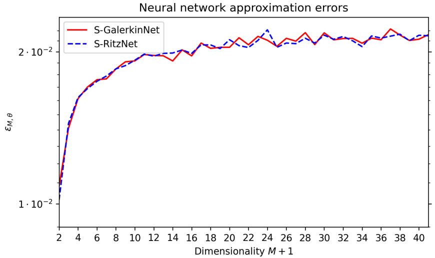
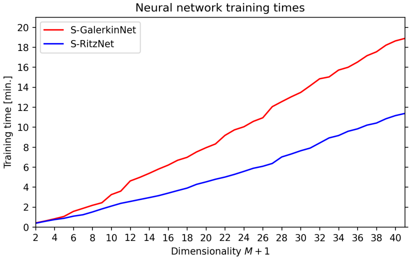
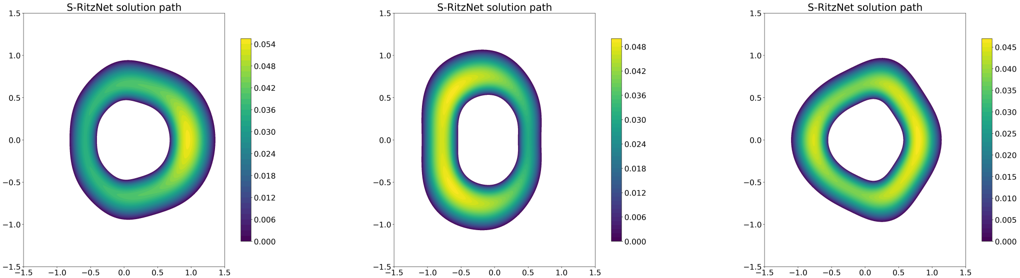


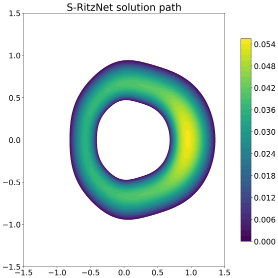


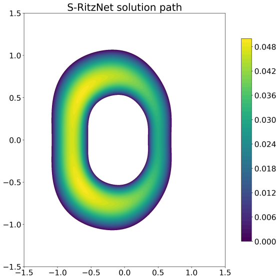


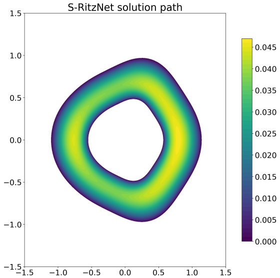
Numerical experiment: random domain problem

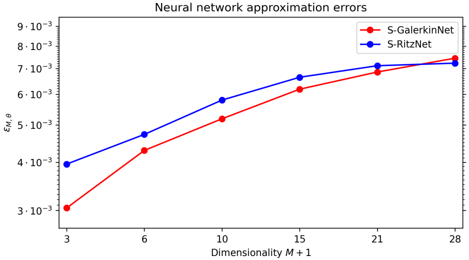


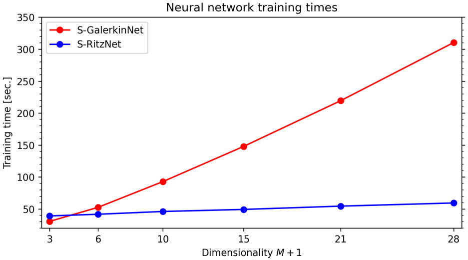
Numerical experiment: random domain problem











Thank you for your attention!

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Appendix: strong Galerkin projection

- Find projection of solution on $L^{2;(M)}(\Gamma;\mathbb{R}) \otimes H_0^2(\mathcal{D};\mathbb{R})$
- Truncated strong residual

$$\mathcal{R}_{\bar{u}^{(M)}}(y,x) := \nabla \cdot \left(\bar{a}^{(M)}(y,x)\nabla\bar{u}^{(M)}(y,x)\right) + \bar{f}^{(M)}(y,x)$$

Strong form stochastic Galerkin approximation

For the truncated deterministic representation $\bar{f}^{(M)} \in L^{2;(M)}(\Gamma; L^2(\mathcal{D};\mathbb{R}))$ of a given stochastic forcing term, find the spectral coefficients $u_0,\ldots,u_M \in H_0^2(\mathcal{D};\mathbb{R})$ of $\bar{u}^{(M)}(y,x) = \sum_{i=0}^{M} u_i(x)\,p_i(y) \in L^{2;(M)}(\Gamma; H_0^2(\mathcal{D};\mathbb{R}))$ such that for every $k = 0,\ldots,M$

$$\left\langle \mathcal{R}_{\bar{u}^{(M)}}(\cdot,x)\,,\,p_k \right\rangle_{L^2(\Gamma;\mathbb{R})} = 0 \qquad x \in \mathcal{D},$$

$$\left\langle \bar{u}^{(M)}(\cdot,x)\,,\,p_k \right\rangle_{L^2(\Gamma;\mathbb{R})} = 0 \qquad x \in \partial\mathcal{D}.$$

Appendix: weak Galerkin projection

- Substitution of truncated deterministic representations of random fields to approximate bilinear and linear form

Weak form stochastic Galerkin approximation

For the truncated deterministic representation $\bar{f}^{(M)} \in L^{2;(M)}(\Gamma; H^{-1}(\mathcal{D};\mathbb{R}))$ of a given stochastic forcing term, find the spectral coefficients $u_0, \dots, u_M \in H_0^1(\mathcal{D};\mathbb{R})$ of $\bar{u}^{(M)}(y,x) = \sum_{i=0}^{M} u_i(x) p_i(y) \in L^{2;(M)}(\Gamma; H_0^1(\mathcal{D};\mathbb{R}))$ such that

$$\bar{B}^{(M)}(\bar{u}^{(M)}, \bar{v}^{(M)}) = \bar{F}^{(M)}(\bar{v}^{(M)}) \text{ for all } \bar{v}^{(M)} \in L^{2;(M)}(\Gamma; H_0^1(\mathcal{D};\mathbb{R})).$$

- Stochastic Ritz approach: Solution u^* is given as

$$u^* = \operatorname*{arg\,min}_{\bar{u}^{(M)} \in L^{2;(M)}(\Gamma; H_0^1(\mathcal{D};\mathbb{R}))} \frac{1}{2} \bar{B}^{(M)}(\bar{u}^{(M)}, \bar{u}^{(M)}) - \bar{F}^{(M)}(\bar{u}^{(M)}) =: \bar{\mathcal{E}}^{(M)}(\bar{u}^{(M)})$$

Appendix: S-GalerkinNet training strategy

- Recap: $\mathcal{R}_{\bar{u}_\theta^{(M)}}(y,x) = \nabla \cdot \left(\left(\sum_{i=0}^{M} a_i(x) p_i(y) \right) \nabla \left(\sum_{j=0}^{M} u_{j;\theta}^{\mathrm{SG}}(x) p_j(y) \right) \right) + \sum_{k=0}^{M} f_k(x) p_k(y)$
- Goal: find $\theta^* \in \Theta^{\mathrm{SG}}$: $\left\langle \mathcal{R}_{\bar{u}_{\theta^*}^{(M)}}(\cdot, x), p_k \right\rangle_{L^2(\Gamma;\mathbb{R})} \overset{!}{=} 0$, for every $k = 0, \dots, M$, $x \in \mathcal{D}$

Theorem

For smooth activation functions and every $\theta \in \Theta^{\mathrm{SG}}$ and $k = 0, \dots, M$, the mapping $x \mapsto \left\langle \mathcal{R}_{\bar{u}_\theta^{(M)}}(\cdot, x), p_k \right\rangle_{L^2(\Gamma;\mathbb{R})}$ is a member of $L^2(\mathcal{D};\mathbb{R})$, i.e. $\int_{\mathcal{D}} \langle \mathcal{R}_{\mathcal{U}_\theta^{\mathrm{SG}}}(\cdot, x), p_k \rangle_{L^2(\Gamma;\mathbb{R})}^2 \, d\lambda(x) < \infty$.

- Unsupervised loss function of least-squares type:

$$\mathbf{L}^{\mathrm{SG}}(x; \mathcal{N}_\theta^{\mathrm{SG}}) := \frac{1}{M+1} \sum_{k=0}^{M} \left\langle \mathcal{R}_{\mathcal{U}_\theta^{\mathrm{SG}}}(\cdot, x), p_k \right\rangle_{L^2(\Gamma;\mathbb{R})}^2$$

- Training of S-GalerkinNet: Find a set of parameters $\theta^* \in \Theta^{\mathrm{SG}}$, such that

$$\lambda(\mathcal{D}) \frac{1}{n} \sum_{i=1}^{n} \mathbf{L}^{\mathrm{SG}}(x_i; \mathcal{N}_{\theta^*}^{\mathrm{SG}}) \overset{!}{=} 0$$

Appendix: S-RitzNet training strategy

- Recap:

$$\bar{\mathcal{E}}^{(M)}(\bar{u}_\theta^{(M)}) = \int_{\mathcal{D}} \int_{\Gamma} \frac{1}{2} \bar{a}^{(M)}(y,x) \, \|\nabla \bar{u}_\theta^{(M)}(y,x)\|_2^2 - \bar{f}^{(M)}(y,x) \, \bar{u}_\theta^{(M)}(y,x) \, d\mu(y) \, d\lambda(x)$$

- Goal: find $\theta^* \in \Theta^{\mathrm{SR}}$: $\bar{u}_{\theta^*}^{(M)} = \underset{\bar{u}_\theta^{(M)}}{\arg\min} \, \bar{\mathcal{E}}^{(M)}(\bar{u}_\theta^{(M)})$
- Unsupervised loss function naturally given:

$$\mathbf{L}^{\mathrm{SR}}(x; \mathcal{N}_\theta^{\mathrm{SR}}) = \int_{\Gamma} \frac{1}{2} \bar{a}^{(M)}(y,x) \, \|\nabla \bar{u}_\theta^{(M)}(y,x)\|_2^2 - \bar{f}^{(M)}(y,x) \, \bar{u}_\theta^{(M)}(y,x) \, d\mu(y)$$

- Training of S-RitzNet: Find a set of parameters $\theta^* \in \Theta^{\mathrm{SR}}$, such that

$$\lambda(\mathcal{D}) \frac{1}{n} \sum_{i=1}^{n} \mathbf{L}^{\mathrm{SR}}(x_i; \mathcal{N}_{\theta^*}^{\mathrm{SR}}) \to \min!$$

Appendix: architectures S-GalerkinNet and S-RitzNet